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SELF-SIMILAR SOLUTIONS DESCRIBING UNSTEADY THERMOCAPILLARY FLUID FLOWS†

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Unsteady thermocapillary flows in thin layers and layers of infinite thickness with non-uniform heating of the free boundary are investigated at high Marangoni numbers. In the plane and axially symmetric cases, self-similar solutions of the non-linear boundary-layer equations are constructed and asymptotic formulae are presented. It is shown that the self-similar solutions may be non-unique for certain values of the parameters of the problem. The branching points are calculated numerically and the branched solutions are investigated.

Flows in thin layers, bounded by two solid walls, have been considered previously [1]. Steady flows with a solid and a free boundary have been studied in [2] and unsteady self-similar solutions have been constructed in the plane case [3] for Marangoni boundary layers close to a free boundary.

1. The problem of the unsteady thermocapillary flow of an incompressible fluid in a thin layer bounded by a solid wall S and a free boundary Γ is considered. A non-zero temperature gradient is specified on the free boundary

$$\partial \mathbf{v} / \partial t+(\mathbf{v},\nabla)\mathbf{v}=-\rho^{-1}\nabla p+\upsilon\Delta\mathbf{v}+\mathbf{g} \tag{1.1}$$

$$\partial T / \partial t+\mathbf{v}\nabla T=\chi\Delta T,\quad \operatorname{div}\mathbf{v}=0$$

$$p=2\nu\rho\mathbf{n}\Pi\mathbf{n}-\sigma(k_1+k_2)+p_*,\quad (x,y,z)\in\Gamma \tag{1.2}$$

$$2\nu\rho[\Pi\mathbf{n}-(\mathbf{n}\Pi\mathbf{n})\mathbf{n}]=\nabla_\Gamma\sigma,\quad T=T_\Gamma,\quad (x,y,z)\in\Gamma$$

$$\partial f / \partial t+\mathbf{v}\nabla f=0,\quad (x,y,z)\in\Gamma;\quad \mathbf{v}=0,\quad T=T_S,\quad (x,y,z)\in S$$

Here, $\mathbf{v} = (\upsilon_x, \upsilon_y, \upsilon_z)$ is the velocity vector, $\mathbf{g} = (0, 0, -g)$, g is the acceleration due to gravity, $\mathbf{h}$ is the unit vector of the outward normal to the free boundary Γ, Π is the rate of strain tensor, k_1 and k_2 are the principal curvatures of the surface Γ, p_* is the specified pressure of Γ, $\nabla_1 = \nabla - (\mathbf{n}\nabla)\mathbf{n}$ is the gradient along $\Gamma, f(x, y, z) = 0$ is the equation of the free surface in implicit form, $\sigma = \sigma_0 - |\sigma_T|/(T - T_*)$ is the coefficient of surface tension where σ_0, σ_T, T_* are unknown constants ($\sigma_T < 0$), and T_Γ and T_S are specified values of the temperature on Γ and S. Initial conditions are not specified as only self-similar solutions are constructed below. It is assumed that the coefficient of kinematic viscosity ν and the thermal conductivity χ are small.

When the free boundary is non-uniformly heated, shear stresses occur on it as a consequence of the thermocapillary effect which, when $\nu \to 0$, leads to the formation of non-linear boundary layers. We will reduce problem (1.1), (1.2) to a dimensionless form by introducing the characteristics scales of length $L = \sqrt{(\sigma_0/(\rho g))}$, velocity $U = (\sigma_T^2 A^2 L \sigma^{-2} \nu^{-1})^{1/3}$, pressure $P = \rho U^2$ and time L/U, where A is the characteristic scale of the temperature gradient and, then, introduce a small parameter $\varepsilon = M^{-1/3}$, where $M = |\sigma_T| L^2 A \rho^{-1} \nu^{-2}$ is the Marangoni number, which takes large values. Note that small values of the coefficient of kinematic viscosity ν or large values of the temperature gradient A correspond to small values of ε. We now introduce the parameter $\lambda = |\sigma_T| AL | \sigma_0$, which arises in changing to dimensionless variables in the dynamic boundary condition for the normal stresses at the free surface and use the notation $p_*' = p_*/(\rho U^2)$.

We next consider unsteady flow in a thin layer with a thickness of the order of ε which is bounded below by a solid boundary and, above, by a free boundary.

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Asymptotic expansions of the solution of problem (1.1), (1.2), when $\varepsilon \to 0$, are constructed in the form

$$\mathbf{v} \sim \mathbf{h}_0 + \varepsilon \mathbf{h}_1 + \ldots, \quad p' \sim q_0 + \varepsilon q_1 + \ldots, \quad (p' = (p + \rho g z)/P) \tag{1.3}$$
$$T \sim \theta_0 + \varepsilon \theta_1 + \ldots, \quad \zeta \sim \varepsilon \zeta_1 + \ldots$$

Here, $z = \zeta(x, y, t)$ is the equation of the free boundary.

Boundary-value problems for the leading and higher terms of the asymptotic series (1.3) are found by applying the boundary layer method to system (1.1), (1.2). Let the solid boundary be the plane $z = 0$, on which the origin of the cylindrical system of coordinates r, θ, z is placed. Next, an axially symmetric solution is constructed for which there is no angular velocity component, that is, $v_\theta = 0$ and $\mathbf{v}, p', T, \zeta$ are independent of the coordinate θ. The components of the vector $\mathbf{h}_k$ are denoted by h_{rk}, h_{zk}. We substitute series (1.3) into system (1.1), (1.2) and introduce the strain transformation $z = \varepsilon s$. On equating the coefficients of ε^{-1} and ε to zero, we find that $h_{z0} = 0$ and that h_{r0}, h_{z1} satisfy the Prandtl boundary-layer equations

$$\partial h_{r0}/\partial t + h_{r0} \partial h_{r0}/\partial r + h_{z1} \partial h_{r0}/\partial s = \partial^2 h_{r0}/\partial s^2 - \partial q_0/\partial r \tag{1.4}$$
$$\partial q_0/\partial s = 0, \quad \partial(r h_{r0})/\partial r + \partial(r h_{z1})/\partial s = 0$$

with the boundary conditions

$$\partial h_{r0}/\partial s = -\partial T_\Gamma/\partial r, \quad \partial \zeta_1/\partial t + h_{r0} \partial \zeta_1/\partial t = h_{z1} \quad (s = \zeta_1(r, t))$$
$$h_{r0} = h_{z1} = 0 \quad (s = 0)$$

The boundary conditions when $s = \zeta_1$ are the dynamic condition for the shear stresses and the kinematic condition at the free boundary.

We next supplement system (1.4) with an equation which is obtained when satisfying the dynamic boundary condition for the normal stresses at the free boundary. We shall consider the case when the parameter λ is of the order of ε^2, that is, $\lambda = \lambda_0 \varepsilon^2$. The boundary condition for the normal stresses now reduces to the relation

$$\lambda_0 q_0 = \zeta_1 + \lambda_0 p_*' - \partial^2 \zeta_1/\partial r^2 - r^{-1} \partial \zeta_1/\partial r$$

In the case when $\lambda \ll \varepsilon^2$, the surface $s = \zeta_1(r, t)$ satisfies the latter equation when $\lambda_0 = 0$ which is integrated separately from system (1.4). The case when $\lambda \gg \varepsilon^2$ is not considered.

We will now construct the self-similar solution of system (1.4) subject to the condition that the temperature gradient in Γ depends on the variables r and t as given by the power law $\partial T_\Gamma/\partial r = \tau r t^{1/2}$, and we will represent the functions h_{r0}, h_{z1}, q_0 in the form

$$h_{r0} = r t^{-1} F'(\xi), \quad h_{z1} = -2t^{-1/2} F(\xi), \quad \partial q_0/\partial r = -q r t^{-2}, \quad \xi = s t^{-1/2}$$

The boundary conditions on the free boundary are satisfied if

$$\zeta_1 = h\sqrt{t}, \quad \lambda_0 P_*' = \lambda_0 q r^2 t^{-2}/2 - h\sqrt{t} + c(t)$$

where $c(t)$ is an arbitrary function of time, and h and q are parameters. It is obvious that the free boundary $z = \varepsilon h \sqrt{(t)} + O(\varepsilon^2)$ moves away from the solid wall with a velocity $\varepsilon h/(2\sqrt{t}) + O(\varepsilon^2)$. For the function $F(\zeta)$, we derive the boundary-value problem

$$F''' = F'^2 - 2FF'' - F' - \xi F''/2 - q$$
$$F''(h) = -\tau, \quad F(h) = -h/4, \quad F(0) = F'(0) = 0 \tag{1.5}$$

from system (1.14).

Note that one of the boundary conditions serves to determine the unknown constant q. When $\tau > 0$, the shear stresses at the free boundary are directed towards the axis of symmetry and in the opposite direction when $\tau < 0$.

Problem (1.5) was integrated numerically using the Runge–Kutta method. When h = const for $\tau \in (-4, 4)$, graphs of $q = q(\tau)$ are represented in Fig. 1 by curves 1 and 2, which correspond to the values $h = 2$ and $h = 1$. Note that, when $h = 1$ for values of τ in the range $\tau_1 < \tau < \tau_2$ ($\tau_1 = -1.465$ and $\tau_2 = 1.198$), there is only a flow zone for which $h_{x0} > 0$ in the velocity profile. When $\tau = \tau_2$, the shear stresses on the solid wall vanish and, when $\tau > \tau_2$, a countercurrent zone arises close to this boundary. When $\tau = \tau_1$, the velocity at the free boundary vanishes and, when $\tau < \tau_1$, a counterflow zone arises close to Γ.

Numerical calculations were carried out to determine the dependence of the amplitude of the pressure gradient q on the thickness of the layer h for fixed values of the parameter τ. The calculations show that there is just a single solution for fixed values of h in the range (0, 10) for $\tau \leqslant \tau_* = -0.1695$. When $\tau > \tau_*$, problem (1.5) can have one, two or three solutions depending on the values of the parameter h.

We shall now consider the case when $\tau = -1$ for which curve 1 in Fig. 2 represents the relation $q(h)$. Calculations were carried out for $0 < h \leqslant 10$. A single solution is found for each h. As h increases, the curve $q(h)$ approaches the straight line $q = 2$.

We note three cases. When $0 < h < 1.429$, the velocity profile has a single flow zone ($h_{x0} > 0$). When $h = 1.429$, the velocity on the free boundary vanishes and a counterflow zone arises close to the free boundary in the range $1.429 < h < 4.720$. When $h = 4.720$, the shear stresses on the solid wall vanish. When $h > 4.720$, counterflow zones develop close to the boundaries of the layer when there is a flow zone between them.

We will now consider the case when $\tau = 0$. The relation $q(h)$ is represented by curves 2 and 4 in Fig. 2. There are no shear stresses at the free boundary. When $h < 5.164$, only a single solution is found numerically which belongs to curve 2. When $h > 5.164$, three solutions were found (one belongs to curve 2 and two to curve 4). When $h = 5.164$, the two solutions belonging to curve 4 approach the straight line $q = 2$ and, moreover, the two curves initially intersect this straight line, then reach a maximum and subsequently decrease, approaching this straight line from above.

Solutions corresponding to curve 2 in the range $h \in (0, h_1)$, where $h_1 = 2.585$, only have a single flow zone, the shear stresses on the solid wall vanish when $h = h_1$ and a counterflow develops close to the wall when $h > h_1$.

Curve 4 consists of lower and upper branches which merge when $h = h_* = 5.164$. On the upper branch of this curve when $h > 5.171$, the velocity profile has a flow zone close to the free boundary and a counterflow zone close to the solid wall. When $h = 5.171$, the shear stress on the solid wall vanishes, the counterflow zone disappears and the velocity on the free boundary simultaneously vanishes. On moving to the left along the upper branch of curve 4 from the point $h = 5.171$, reaching the "tip" when $h = h_*$, passing onto the lower branch of this curve and reaching the point with the coordinate $h = 5.184$, we find that the velocity profile here has a flow zone close to the wall and a counterflow zone close to the free boundary. When $h = 5.184$, the shear stress on the wall vanishes and, when $h > 5.184$, two counterflow zones (one close to the wall and the other close to the free boundary) develop in the velocity profile with a flow zone between them.

The graph of $q(h)$, for the case when $\tau = 1$, is represented by curves 3 and 5 in Fig. 2. The extreme left-hand point of curve 5, at which the upper and lower branches merge, corresponds to the value $h = 6.853$. Unlike the case when $\tau = -1$, two flow zones and two counterflow zones appear on the lower branch of curve 5 when $h > 6.672$.

2. We shall now consider the plane problem of the thermocapillary flow of a fluid in a thin layer with a thickness of the order of ε, bounded below by a solid wall and above by a free boundary. We introduce a Cartesian system of coordinates with its origin on the solid wall. The equations and boundary conditions for h_{x0}, h_{z1} (the components of the vectors $\mathbf{h}_0$, $\mathbf{h}_1$) and the functions q_0, ζ_1 are obtained from (1.4) by

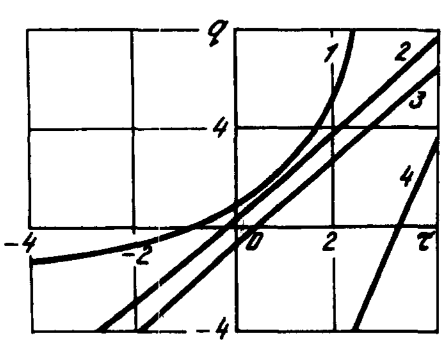

Fig. 1.

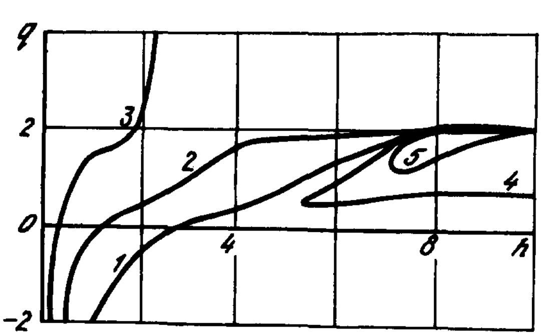

Fig. 2.

replacing h_{r0} and the coordinate r by h_{x0} and x, respectively, with the equation of continuity for the plane problem and the dynamic condition in the free boundary

$$\lambda_0 q_0 = \zeta_1 - \partial^2\zeta_1/\partial x^2 + \lambda_0 p'_*$$

A self-similar solution exists when $\partial T_\Gamma/\partial x = \tau x t^{-1/2}$ and is written in the form

$$h_{x0} = -xt^{-1}\psi'(\xi),\quad h_{z1} = t^{-1/2}\psi(\xi),\quad \partial q_0/\partial x = qxt^{-2},\quad \xi = st^{-1/2}$$

The boundary conditions on the free boundary are satisfied if ζ_1 and p'_* are chosen in the same manner as in the axially symmetric case.

The function $\psi(\xi)$ is determined from the boundary-value problem

$$\psi''' + (\xi/2 - \psi)\psi'' + \psi'^2 = q \tag{2.1}$$

$$\psi(0) = \psi'(0) = 0,\quad \psi''(h) = \tau,\quad \psi(h) = h/2$$

Results of numerical calculations of the relation $q(\tau)$ are represented by curves 3 and 4 in Fig. 1 for $h = 1$ and $h = 0.3$, respectively. Note that, when $h = 1$ in the range $-2.909 < \tau < 2.221$, the velocity profile has just a single flow zone ($h_{x0} > 0$). When $\tau = -2.909$, the velocity at the free boundary vanishes and, when $\tau < -2.909$, a counterflow zone appears close to Γ. When $\tau = 2.221$, the shear stresses on the solid wall S vanish and, when $\tau > 2.221$, a countercurrent zone appears close to S.

Numerical calculations were also carried out in order to determine how the pressure gradient depends on the thickness of the layer h for fixed τ. The relation $q(h)$ is shown in Fig. 3 for $\tau = 0$ (the solid curves) and $\tau = 1$ (the dashed curves). Two branches of the solutions are found for $\tau = 1$. For one branch, solutions are only obtained in the range $0 < h < h_1 = 2.251$ while, for the second branch, they are only found when $h \leqslant h_2 = 6.718$. For values of h close to h_1 and h_2, problem (2.1) has two solutions, for one of which the function $q(h)$ rapidly increases as the parameter h decreases from the values of h_1 and h_2.

For small values of h, on expanding the function $\psi(\xi)$ in series in powers of ξ and retaining three terms, we find

$$q = \tfrac{3}{2}(-h^{-2} + h/\tau)[1 + o(1)]\quad (h \to 0)$$

When $h = 0.1$ and $\tau = 1$, the three significant figures are identical in the case of the numerical and the asymptotic values.

Two branches of solutions are also found when $\tau = 0$. One branch is calculated in the range $0 < h \leqslant h_1 = 2.644$ and the other when $h > h_2 = 4.406$. For h close to h_1 and h_2, but outside the range (h_1, h_2), two solutions were calculated for each case. No bounded numerical solutions were found when $h \in (h_1, h_2)$. Now, unlike in the case when $\tau = 1$, solutions were found for any $h > h_2$. Note that calculations were also carried out for negative τ, such as $\tau = -0.3$, for example, and the qualitative behaviour of the function $q(h)$ was similar to that in the case when $\tau > 0$.

3. We shall now consider the unsteady, thermocapillary, axially symmetric, spatial flow of a fluid in an unbounded domain (a layer of infinite thickness) when there is non-uniform heating of the free

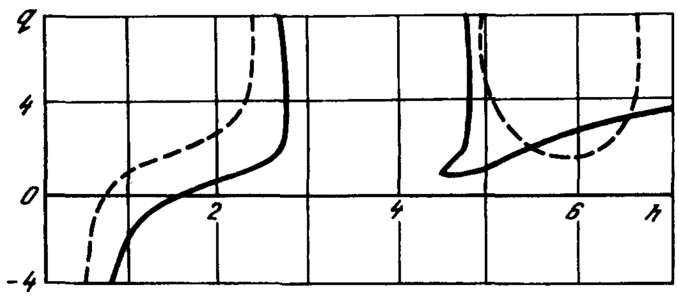

Fig. 3.

boundary. Instead of the conditions on the solid wall in the boundary conditions (1.2), the condition at infinity $\mathbf{v} \to \mathbf{U}_0$ ($z \to \infty$) is now imposed. Here, the origin of the cylindrical system of coordinates is placed in the free boundary and the z-axis is directed into the half-space occupied by the fluid. Unlike (1.3), we now construct the asymptotic expansion of the solution of the problem when $\varepsilon \to 0$ in the form

$$\mathbf{v} \sim \mathbf{h}_0 + \mathbf{v}_0 + \varepsilon(\mathbf{h}_1 + \mathbf{v}_1) + \ldots, \quad p \sim p_0 + \varepsilon(p_1 + q_1) + \ldots$$

We shall denote the boundary-layer domain by D_Γ. Then, $\mathbf{h}_0$, $\mathbf{h}_1$, $q_1, \ldots$ are functions of the type of solutions of the boundary-layer problem in D_Γ. Outside D_Γ, the functions $\mathbf{v}_k, p_k$ determine the solution of the problem, while $\mathbf{v}_0, p_0$ describe inviscid flow and satisfy Euler's equations.

We shall assume that the vector $\mathbf{v}_0$ has the components $(U_0(r, t), 0, 0)$. It then follows from Euler's equations that

$$\partial U_0 / \partial t + U_0 \partial U_0 / \partial r = -\partial p_0 / \partial r$$

This equation has the solution

$$U_0 = rt^{-1}U_\infty, \quad p_0 = r^2 t^{-2}(U_\infty - U_\infty^2)/2 + c(t)$$

where $U_\infty = \text{const}$ and $c(t)$ is an arbitrary function of time. In the case of this solution, the free boundary (when the viscosity is zero) turns out to be the plane $z = 0$ if one puts $p_* = p_*^{(0)} + \varepsilon p_*^{(1)} + \ldots$ and $p_*^{(0)}$ is specified as in the preceding axially symmetric case (taking account of the relation $h = 0$).

We will now construct a self-similar solution, assuming that the vector $\mathbf{h}_0$ is independent of the coordinate θ. The longitudinal components u_{r0}, $u_{\theta 0}$ of the vector $\mathbf{u}_0 = \mathbf{h}_0 + \mathbf{v}_0$ and the transverse component u_{z1} of the vector $\mathbf{h}_1 + \mathbf{v}_1$ ($u_{z0} = 0$) in the boundary layer satisfy the Prandtl spatial conditions. We shall only present the equation for the component $u_{\theta 0}$

$$\partial u_{\theta 0} / \partial t + u_{r0} \partial u_{\theta 0} / \partial r + u_{z1} \partial u_{\theta 0} / \partial z + u_{\theta 0} u_{r0} / r = \partial^2 u_{\theta 0} / \partial s^2 - \partial p_0 / \partial \theta \tag{3.1}$$

The equation for u_{r0} is obtained from (3.1) by replacing the derivatives of $u_{\theta 0}$ with the derivatives of u_{r0}, while the terms $u_{\theta 0}u_{r0}$ and $\partial p_0/\partial\theta$ are replaced by $-u_\theta{}^2$ and $\partial p_0/\partial r$, respectively.

The boundary conditions reduce to the form

$$\partial u_{r0} / \partial s = -\partial T_\Gamma / \partial r, \quad \partial u_{\theta 0} / \partial s = 0, \quad u_{z1} = 0 \quad (s = z, \ \varepsilon = 0)$$
$$u_{r0} \to U_0, \quad u_{\theta 0} \to 0 \quad (s \to \infty)$$

It is assumed that the temperature gradient on Γ is independent of the coordinate θ.

It can be shown that $q_1 = 0$ and that the deformation of the free boundary is of the order of $O(\varepsilon^2)$ if the function $p_*^{(1)}$ at the free surface is specified in the appropriate manner. Only a two-dimensional Prandtl system with boundary conditions on a plane free boundary has been studied previously [3].

We shall define the function $T_\Gamma(r, t)$ by the relation $\partial T_\Gamma/\partial r = \tau r t^{-1/2}$ and represent the solution of problem (3.1) in the form

$$u_{r0} = rt^{-1}F'(\xi), \quad u_{\theta 0} = rt^{-1}G(\xi), \quad u_{z1} = -2t^{-1/2}F(\xi), \quad \xi = s/\sqrt{t}$$

The functions F and G are found by solving the boundary-value problem

$$\begin{aligned} &F''' = F'^2 - 2FF'' - F' - \xi F''/2 - G^2 + U_\infty - U_\infty^2 \\ &G'' = 2(F'G - FG') - G - \xi G'/2 \\ &F''(0) = -\tau, \quad G'(0) = F(0) = G(\infty) = 0, \quad F'(\infty) = U_\infty \end{aligned} \tag{3.2}$$

Numerical calculation of system (3.2) leads to the conclusion that several solutions arise depending on the magnitude and direction of the temperature gradient. When $\tau < \tau_*$, two symmetric solutions with rotation are found. Four solutions are calculated in the range (τ_*, τ_b), of which two solutions are with rotation and two solutions are without rotation. When $\tau > \tau_b$, only two solutions without rotation are found.

We shall initially consider a flow in which there is no angular component of the velocity ($u_{\theta 0} = 0$) and put $G = 0$ in (3.2). We introduce the notation $U_\Gamma^{90} = F'(0)$ and define U_∞ and U_Γ as the "amplitudes" of the velocity at infinity and the velocity at the free boundary. We now present an analysis of the numerical results when $U_\infty = 1$. Curve 1 ($A_0A_1A_*A_bA_2A_3$) in Fig. 4 represents U_Γ as a function of τ. A solution of problem (3.2) is only found when $\tau_* \leqslant \tau < \infty$ and, when $\tau < \tau_*$, no bounded numerical solutions are obtained. Note that $\tau_* = -1.060$. Two solutions are calculated for each $\tau > \tau_*$. On the A_0A_1 branch, the velocity at the free boundary is positive and the velocity profile is monotonic. At point A_1, we show the values $\tau_1 = -1.035$ and $U_\Gamma = 0$. When $\tau = 0$, the solution is trivial: $u_{r0} = 1$ and, when $\tau > 0$, the velocity in the boundary layer decreases monotonically as ξ increases while, when $\tau < 0$, the velocity increases monotonically. On the $A_1A_*A_2$ branch, the values of τ vary in the range (τ_*, τ_2), where $\tau_2 = 0.619$. There is a counterflow zone ($u_{r0} < 0$) close to the free boundary and, at large ξ, a flow zone. For values corresponding to the A_2A_3 branch, there are two flow zones, one of which is located close to the free boundary and another at large ξ, and there is a flow zone between them.

The analysis is carried out in a similar manner when $U_\infty = 0$. Curve 2 in Fig. 4 represents $U_\Gamma(\tau)$. Note that solutions with $u_{\theta 0} = 0$ are not found when $\tau < \tau_* = -0.054$. When $\tau = 0$, a single non-trivial solution and a single trivial solution $u_{\theta 0} = 0$ are found.

We now present the asymptotic solutions, as $\tau \to \infty$, for the upper branches of curves 1 and 2 in Fig. 4. We introduce the function $f(\eta) = \tau^{-1/3}/(F - U_\infty \xi)$, where $\eta = \tau^{1/3}\xi$ and note that $f(\infty) = 0$. We represent $f(\eta)$ and U_Γ in the form of the asymptotic series

$$f = f_0(\eta) + \tau^{-2/3} f_1(\eta) + O(\tau^{-4/3}),\quad U_\Gamma = \tau^{2/3} f_0'(0) + U_\infty + f_1'(0) + O(\tau^{-2/3}) \quad (\tau \to \infty)$$

The functions f_0 and f_1 are determined from the boundary-value problems

$$f_0''' + 2f_0 f_0'' - f_0'^2 = 0,\quad f_0(0) = f_0'(\infty) = f_0''(0) + 1 = 0$$

$$f_1''' + 2f_0 f_1'' - 2f_1 f_0'' - 2f_0' f_1' = (2U_\infty - 1) f_0' - \eta(1/2 + 2U_\infty) f_0'',\quad f_1(0) = f_0''(0) = f_1'(\infty) = 0$$

Numerical calculations lead to the values $f_0'(0) = 0.8987$ and $f_1'(0) = -0.6322$ when $U_\infty = 1$ and, also, $f_1'(0) = 0.1542$ for the same value of $f_1'(0)$ for $U_\infty = 0$. The dashed curve in Fig. 4 close to the upper branch of curve 1 for positive values of τ represents the asymptotic solution for $U_\infty = 1$.

We shall now consider the solution of system (3.2) when $\tau < \tau_*$. In this case, $u_{\theta 0} \neq 0$ (a flow appears with a rotation around the z-axis). A branch of solutions is found numerically, which, when $U_\infty = 1$, is represented by the curve A_bA_∞ in Fig. 4 (and the corresponding branch for $U_\infty = 0$). Note that the point A_b with the coordinate $\tau = \tau_b = -0.895$ turns out to be a branch point (when $U_\infty = 0$, the value of τ_b is equal to -0.038) and $\tau_b > \tau_*$. When $\tau \to \tau_b - 0$, the angular component of the velocity $u_{\theta 0}$ tends to zero and vanishes when $\tau = \tau_b$. Two symmetric solutions (u_{r0}, $\pm u_{\theta 0}$, u_{z1}) appear along the $A_\infty A_b$ branch, one of which is obtained from the other by reversing the direction of the angular component of the velocity.

We will now construct the asymptotic solution of problem (3.2) as $\tau \to -\infty$. We introduce the variable $\eta_1 = (-\tau)^{1/3}$ and represent the functions F, G, U_Γ in the form of the series

$$F = (-\tau)^{1/3} F_0(\eta_1) + (-\tau)^{-1/3}[\eta_1 U_\infty + F_1(\eta_1)] + O((-\tau)^{-1})$$

$$G = (-\tau)^{2/3} G_0(\eta_1) + G_1(\eta_1)] + O((-\tau)^{-2/3})$$

$$U_\Gamma = (-\tau)^{2/3} F_0'(0) + U_\infty + F_1'(0) + O(1-\tau)^{-2/3}) \quad (\tau \to \infty)$$

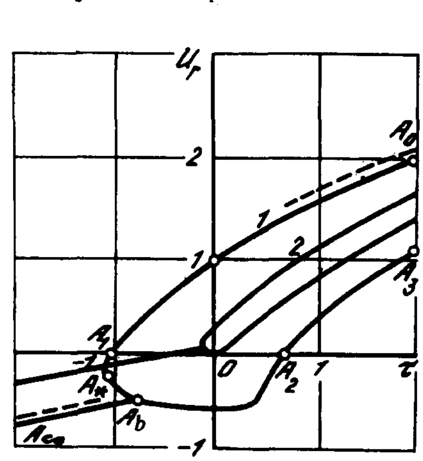

Fig. 4.

The functions F_0, F_1, G_0, G_1 are determined by solving boundary-value problems, which are not given here on account of their complexity. We give the values $F_0'(0) = -0.5181$, $F_1'(0) = -0.6882$, $G_0(0) = 1.0659$, $G_1(0) = 0.0085$ when $U_\infty = 1$. The dashed curve close to the $A_\infty A_b$ branch in Fig. 4 represents the asymptotic solution when $U_\infty = 1$.

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